Evaluate All Possible Subgroups of a Group of Order 30 and 42 By Using Sylow's Theorem

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Abstract:

In this work, we will discuss the concept of group, order of a group, along with the associated notions of automorphisms group of the dihedral groups and split extensions of groups. This work is a generalization of the Sylow's Theorems. Then we find all the groups of order 30 and 42 which will give us a practical knowledge to see the applications of the Sylow's Theorems. For this, we also have used some known results of Semi-direct Product of groups, some important definitions as like as the exact sequences and split extensions of groups and $P-$Sylow's Theorem to obtain our result. Finally, we have found all subgroups of group order 30 and 40 for Abelian and Non-abelian cases.

Key words:

Dihedral group, exact sequences, split extensions of groups, Lagrange's Theorems and P-Sylow's Theorems.

Introduction:

It's not true for any number dividing the order of a group, there exists a subgroup of that order. For example, the group S_4 of even permutations on the set {1, 2, 3, 4} has order 12, yet there does not exist a subgroup of order 6. As usual we can use Lagrange's Theorem to evaluate subgroups of group of different orders such as order $2,4,6,8,9,10,12,......$ etc., i.e. whose order not so high (Not higher order groups). But it is not possible to evaluate subgroups of higher order group as like as $30,35,40,45,50,.....$etc by using Lagrange's Theorems. For this case, applying P-Sylow's Theorems we can easily evaluate all possible subgroups of any higher order groups. The Sylow's Theorems is very important part of finite group theory and the classification of finite simple groups [1, 3]. The order of sylow's P-subgroup of a finite group G is P^n, where

n is the multiplicity of P in the order of G and any subgroup of order P^n is a Sylow P-subgroup of G.

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Preliminaries:

Dihedral group:

A dihedral group is the group of symmetries of a regular polygon, which includes rotations and reflections. Dihedral groups are among the simplest examples of finite groups, and they play an important role in group theory, geometry, and chemistry.

Notation of dihedral group: $D_n = \{a, b : a^n = b^2 = (ab)^2 = I\}$

Definition of P-group:

When p is a prime number, then a p-group is a group, all of whose elements have order some power of p. For a finite group, the equivalent definition is that the number of elements in Gis a power of p. In fact, every finite group has subgroups which are p-groups by the Sylow's theorems, in which case they are called Sylow *p*-subgroups.

Definition of Sylow P-subgroup:

If p^k is the highest power of a prime p dividing the order of a finite group G, then a subgroup of G of order p^k is called a Sylow p-subgroup ofG.

Index of a group:

Let G be a group. Let H be a subgroup of G.

The index $[G:H]$ of H in G is the number of left (or right) cosets of G modulo H, or, the number of elements in the left (or right) coset space G/H.

Lagrange's Theorem:

The order of each subgroup of a finite group, is a divisor of the order of the group, Such that $\frac{\circ(G)}{\circ(H)} = k$. i. e the order of H is a divisor of order of G.

Sylow's First Theorem:

Let G be a finite group and p be a prime number. If m is the largest non-negative integer such that p^m is a divisor of $\circ(G)$, then G has a subgroup of order p^m.

Sylow's second theorem:

Let G be a finite group and let p be a prime number such that p is a divisor of $\circ(G)$. Then, all sylow p-subgroups of G are conjugates of one another.

Sylow's third theorem:

Let G be a finite group and p be a prime number such that $p \mid \circ(G)$. Then the number of sylow p-subgroups is of the form $1+mp$, where m is some non-negative integer.

Sylow's Fourth Theorem:

The number of Sylow p-subgroups of a finite group is congruent to $1(\bmod p)$.

Sylow's Fifth Theorem:

The number of Sylow p-subgroups of a finite groups is a divisor of their common index.

Automorphisms group of the dihedral group D_4:

Let $D_4=\{e, x, x^2, x^3, y, yx, yx^2, yx^3\}$ with the defining relation $x^4=y^2=e$, $y^{-1}xy=x^{-1}$, be the dihedral group of order 8.

Now, the conjugate classes of D_4 are :

$\{e\}, \{x^2\}, \{x, x^3\}, \{y, yx, yx^2, yx^3\}$.

So, $D_4/\{e, x^2\}\cong$ to a group of order 4. So, D_4 has 4 inner automorphisms one of which is the identity. Then, let the other 3 inner automorphisms be α, β, γ .Now, if x is fixed by α then $\alpha(e)=e$, $\alpha(x)=x$ and $\alpha(y)=y, yx, yx^2, or\ yx^3$ .But $\alpha(y)\neq y$, for if $\alpha(y)=y$ then $\alpha = Id$, which is not possible. Then, let $\alpha(y)=yx^2$ and hence $\alpha(yx)=\alpha(y)\alpha(x)=yx^3$ and therefore, $\alpha^2 = Id$ .Next, if y is fixed by β then $\beta(e)=e$, $\beta(y)=y$ and $\beta(x)= x^{-1}$ $\beta(yx)=\beta(y)\beta(x)= yx^{-1}$ and $\beta^2 = Id$. Then $\gamma(e)=e$ and $\gamma(yx)=yx$ and $\gamma(x)=x^{-1}$, $\gamma(y)=yx^2$ and $\gamma^2=Id$ .Hence, we have $\gamma^2=\beta^2=\alpha^2=Id$ and also we have $\alpha\beta=\beta\alpha=\gamma$ and $\alpha\gamma=\gamma\alpha$. Therefore inner $Aut(D_4)=\{Id, \alpha, \beta, \beta\alpha\}\cong C_2\times C_2$ with $\alpha^2 = \beta^2 = Id$ and $\alpha\beta = \beta\alpha$.

Now, we consider the mapping $f : D_4 \to D_4$ With $f(e)=e$ and $f(x)=x \, or \, x^3$.So, let $f(x)=x$ and assume that $g(x)=\alpha \, f(x)$then $g(x)=\alpha(x)=x$ and $g(y)\neq x^2$ for x^2 is a central element and hence $g(y)=y, \, yx, \, yx^2, \, or \, yx^3$.

If $g(y)= y,$ then $g=Id$, and hence $g(y)\neq y$

If $g(y)=yx^2$ then $g=\alpha$ and hence $g(y)\neq yx^2$.

If $g(y)=yx$ then $g(yx)=yx^2$ and $g^4=Id$.

Then, we have, $\beta g\beta = g^{-1}$ with $g^4 = \beta^2 = Id$,and also $\gamma g\gamma = g^{-1}$, with $g^4 = \gamma^2 = Id$.

Therefore, $Aut(D_4)\cong \{\beta, g\}$ with $g^4 = \beta^2 = Id$ and $\beta^{-1} g\beta = g^{-1}$.

Automorphisms group of the dihedral group D_6:

Let $D_6 = \{e, x, x^2, x^3, x^4, x^{5,}, y, yx, yx^2, yx^3, yx^4, yx^5\}$

With defining relation $x^6 = y^2 = e$ and $y^{-1}xy = x^{-1}$, be a dihedral group of order 12.

Now, the conjugate classes are :

$\{e\}, \{x, x^5\}, \{x^2, x^4\}, \{x^3\}, \{y, yx^2, yx^4\}, \{ yx, yx^3, yx^5 \}.$

So, $D_6/\{e, x^3\}\cong$ to a group of order 6. Then D_6 has 6 inner automorphisms one of which is the identity. Let the other inner automorphisms be Y, Z, U, V, T .Now, if x is fixed by Y then $Y(e)=e$, $Z(x)=x$and $Y(y)=yx^2$ and hence $Y(yx)=yx^5$.Then $Y^3 = Id$. Next, if y is fixed by Z then $Z(e)=e$, $Z(y)=$ y, and $Z(x)=x^{-1}$ and $Z(yx)=yx^{-1}$ and then $Z^2=$ Id. Next, if yx is fixed by U then $U(e)=e$, $U(yx)=yx$, and $U(x)=x^{-1}$ and $U(y)=yx^2$. Lastly, if $yx^5$is fixed by T then $T(e)=e$, $T(yx^5)=yx^5$ and $T(x)=x^{-1}$ and $T(yx)=yx^3$ and then $T^2=Id$.

and hence we have, $Y^3 = Z^2 = U^2 = V^3 = T^2 = Id$ and by calculation we have, $Y^2 = V$, $TU = V = ZT$, $UT = Y = TZ$ and hence $Z^{-1}YZ = Y^{-1}$, $U^{-1}VU = V^{-1}$, $T^{-1}YT = Y^{-1}$.

Therefore, inner $Aut(D_6)=\{Z, Y\}\cong D_3 \cong S_3$ with $Z^{-1}YZ = Y^{-1}$ and $Y^3 = Z^2 = Id$.

Now, consider the mapping $S : D_6 \to D_6$

Let $S(e)=e$ then $S(x)=x$ or x^5 and so let $S(x)=x^5$ and put $M=US$ then $M(x)=US(x)=U(x^5)=x$

Now, $M(y)\neq x^3$ for x^3 is a central element.

If $M(y) = y$ then $M = Id$ and hence $M(y) \neq y$

If $M(y) = yx^2$ then $M = U$ and hence $M(y) \neq yx^2$

If $M(y) = yx^4$ then $M = Y$ and hence $M(y) \neq yx^4$

If $M(y) = yx$ then $M(yx) = yx^2$ and $M^6 = Id$ .Now, $MZ = ZM^5$, $MT = TM^5$ and $MU = UM^5$, $Aut(D_6) = \{M, Z\} \cong D_6$ with $M^6 = Z^2 = Id$ and $Z^{-1}MZ = M^{-1}$.

Evaluate all possible subgroups of a group of order 12 where $(G,+) = \{0,1,2,...11, \bmod 12\}$ by using Lagrange's Theorem:

According to the Lagrange's theorem the group has possible subgroup of order 1, 2,3,4,6 and 12.

i) **The subgroup of order 1 is $\{0\}$ itself and it is an improper subgroup.**

ii) **The subgroup of order 12 is the group $\{G,+\}$ itself and it is an improper subgroup.**

iii) **The subgroup of order 2:**

To find a subgroup of order 2, we take an element from the group of order 2.

Here 6 is the element of order 2 as $6+6=12 \equiv 0 \bmod 12$.

Now we consider all the multiplies of 6.

$6 \cdot 0 = 0$

$6 \cdot 1 = 0$

$6 \cdot 2 = 12 \equiv 0 (\bmod 12)$

$6 \cdot 3 = 18 \equiv 6 (\bmod 12)$

$6 \cdot 4 = 24 \equiv 0 (\bmod 12)$

$6 \cdot 5 = 30 \equiv 6 (\bmod 12)$

$6 \cdot 6 = 36 \equiv 0 (\bmod 12)$

$6 \cdot 7 = 42 \equiv 6 (\bmod 12)$

$6 \cdot 8 = 48 \equiv 0 (\bmod 12)$

$6 \cdot 9 = 54 \equiv 6 (\bmod 12)$

$6 \cdot 10 = 60 \equiv 0 (\bmod 12)$

$6 \cdot 11 = 66 \equiv 6 (\bmod 12)$

Therefore the subgroup of order 2 is $(k,+) = \{0,6\}$

iv) To find a subgroup of order 3:

To find a subgroup of order 3, we take an element from the group of order 3.

Here 6 is the element of order 3 as $4+4+4=12 \equiv 0 \ (\bmod 12)$.

Now we consider all the multiplies of 4.

$$4 \cdot 0 = 0$$
$$4 \cdot 1 = 0$$
$$4 \cdot 2 = 8$$
$$4 \cdot 3 = 12 \equiv 0(\text{mod}\, 12)$$
$$4 \cdot 4 = 16 \equiv 4(\text{mod}\, 12)$$
$$4 \cdot 5 = 20 \equiv 8(\text{mod}\, 12)$$
$$4 \cdot 6 = 24 \equiv 0(\text{mod}\, 12)$$
$$4 \cdot 7 = 28 \equiv 4(\text{mod}\, 12)$$
$$4 \cdot 8 = 32 \equiv 8(\text{mod}\, 12)$$
$$4 \cdot 9 = 36 \equiv 0(\text{mod}\, 12)$$
$$4 \cdot 10 = 40 \equiv 4(\text{mod}\, 12)$$
$$4 \cdot 11 = 44 \equiv 8(\text{mod}\, 12)$$

Therefore the subgroup of order 3 is $(L,+) = \{0,4,8\}$.

v) The subgroup of order 4:

To find a subgroup of order 4, we take an element from the group of order 4.

Here 6 is the element of order 4as $3+3+3+3=12 \equiv 0\ (\text{mod}\, 12)$.

Now we consider all the multiplies of 3.

$$3 \cdot 0 = 0$$
$$3 \cdot 1 = 3$$
$$3 \cdot 2 = 6$$
$$3 \cdot 3 = 9$$
$$3 \cdot 4 = 12 \equiv 0(\text{mod}\, 12)$$
$$3 \cdot 5 = 15 \equiv 3(\text{mod}\, 12)$$
$$3 \cdot 6 = 18 \equiv 6(\text{mod}\, 12)$$
$$3 \cdot 7 = 21 \equiv 9(\text{mod}\, 12)$$
$$3 \cdot 8 = 24 \equiv 0(\text{mod}\, 12)$$
$$3 \cdot 9 = 27 \equiv 3(\text{mod}\, 12)$$
$$3 \cdot 10 = 30 \equiv 6(\text{mod}\, 12)$$
$$3 \cdot 11 = 33 \equiv 9(\text{mod}\, 12)$$

Therefore the subgroup of order 2 is $(M,+) = \{0,3,6,9\}$

vi) The subgroup of order 6:

To find a subgroup of order 6, we take an element from the group of order 6.

Here 6 is the element of order 6 as $2+2+2+2+2+2=12\equiv 0 \bmod 12$.

Now we consider all the multiples of 2.

$$
\begin{aligned}
&2\cdot 0=0\\
&2\cdot 1=2\\
&2\cdot 2=4\\
&2\cdot 3=6\\
&2\cdot 4=8\\
&2\cdot 5=10\\
&2\cdot 6=12\equiv 0(\bmod 12)\\
&2\cdot 7=14\equiv 2(\bmod 12)\\
&2\cdot 8=16\equiv 4(\bmod 12)\\
&2\cdot 9=18\equiv 6(\bmod 12)\\
&2\cdot 10=20\equiv 8(\bmod 12)\\
&2\cdot 11=22\equiv 10(\bmod 12)
\end{aligned}
$$

Therefore the subgroup of order 6 is $(N,+)=\{0,2,4,6,8,10\}$.

ALL GROUP OF ORDER 30

Non – Abelian Case

We keep in mind that 30=2.3.5

2- Sylow Subgroups:

The number x of 2-Sylow subgroup of a group G Of order 30 is $x\equiv 1(\bmod ulo 2)$, where x= 1, 3,5,15.

1, 2-Sylow subgroup:

It implies that there is a proper normal subgroup in G which may be called N of order 2. Therefore, $N\cong C_2$ If $N\cong C_2$, then the sequence of group extension $\{e\}\to N\to G\to C_{15}\to\{e\}$

but, (2,15)=1, so the extension splits. Now, $Y:C_{15}\to Aut(C_2)\cong Id$ and (15,2)=1, Where Y is a constant homomorphism and the relation is given by $b^{-1}ab=a^{-1}$ which is a commutative case. So, we exclude this case.

3, 2-Sylow subgroup:

The group G as a permutation on the objects, namely is 3,2-Sylow subgroups. It is a transitive group then the mapping $Y : G \to S_3$ gives that $Y(G) = A_3$ or S_3

i) If $Y(G) = A_3$ then the order of N is 10 and hence $N = C_{10}$ or $N \cong C_5XC_2$ or $N \cong D_5$, Now first two case are exclude for 2-Sylow subgroups of C_{10} and C_5XC_2 are characteristics.

ii) If $Y(G) = S_3$ then the order of N is 5. If $N \cong C_5$ then there is a mapping Z such that $Z : S_3, Aut(C_5) = C_4 = C_2XC_2$. Hence $Z(S_3) = \{e\}$ or C_2 Now, the group extension is given by $\{e\} \to N = C_2XC_2 \to G\ _1\!\to S_3 \to \{e\}$ where the defining relations of S_3 are $a^2 = b^2 = e$ and $b^{-1}ab = a^{-1}$. If ($Z(S_3) = \{e\}$) then $\ker(Z) = S_3$ and so let $N = \{t, u\}$ and $G = \{c, d\}$, where c, d are mapped respectively to a,b . Now $a^2 = b^2 = e$,

$c^2 = e, a = c^{-1}ac, a = b^{-1}ab, b = c^{-1}bc, a = d^{-1}ad, d^{-1}bd = b$ and

$d = a^i b, d^{-1}cd = c^{-1}a^i b^k, a^i, b^k \in C_2 \times C_2 \in Aut(C_5) = D_5$ Now there exists some S $\in$ G such that $s^2 = b^j$ with j=0,1 If $s^2 = b$ the

$a^2 = s^3 = c^2, ac = ca, as = sa, s^{-1}cs = c^{-1}ab, s^2cs^2 = s^{-1}(s^{-1}cs) = ca^{2i}b^{2k} = c$

and $s^2 \in \{a, b\}$ comments with c This implies that $a^{2i}b^{2k} = e$ with 2i=0(mod5)and 2k=0(mod2)which implies that i=0, k=0,1 . If k =o then $s^{-1}cs = c^{-1}$ and if k=1 then $s^{-1}cs = c^{-1}ba$. Note that C_3 generated by a is central because it commutes with every element, put

$f = c^2d^2, f^7 = e, d^2 = f^3$ and $d^{-1}cd = c^{-1}$ and so $\{f, d\}$ generates D_5 but C_3 and D_5 are normal subgroups and $C_3 \cap D_5 = \{e\}$ and hence $G \cong C_3xD_5$ which is a non –abelian group of 30.

5-2 Sylow subgroup:

The normalized of 2-sylow subgroup $N(S_2)$ must have an invariant subgroup of order 2.Now, the order of $N(s_2) = 4$ and So, $N(s_2) \cong C_4$ or $C_2 \times C_2$ or D_2 but $N(s_2) \neq D_2$ because none of them can have an invariant subgroup of order 2. The possibilities are (1) $N(s_2) = C_5$ (2) $C_5x\Delta_{15}$ and so by Burnside's Theorem normal 2-complement exists. This will be abelian, So we exclude this case.

15, 2-Sylow Subgroup:

15, 2-Sylow subgroups imply that the order of $N(S_2)$ is 2

Now, the normal 2- component N of order 15, given that $N \cong C_{15}$ then group extension is given by $\{e\} \to C_{15} \to D \to H \to \{e\}$, where $H \cong C_2$ But (2, 15) =1 and so extension splits.

b1. If $H \cong C_{15}$ then $Y: C_2 \to Aut(C_{15}) \cong C_2 \times C_3$ and $Y(C_2) = \{e\}$

b2. If $Y(C_2) = \{e\}$ then $D \cong C_{15} x C_2$, which is abelian case and hence we drop it

b3. If $Y(C_2) = C_2$ then Y(c)=Z

b4. If Y(c) =Z then $c^{-1}ac = a, c^{-1}bc = b, ab = ba$ and this gives that G≅ $G \cong D_5 \times C_3$, which has already been found.

3-Sylow Subgroup:

The number x of 3-Sylow subgroups of a group G of order 30 is x = 1(modulo 3); where $x = 1$,10.

1, 3-Sylow Subgroups:

Any C_3 is a normal subgroup of G.

The group extension is $\{e\} \to C_3 \to G \to H \to \{e\}$ and H is of

order 10. But (3, 10) =1, so the extension splits. And so $Y: H \to Aut(C_3) \cong C_2$.

(a) Let $H = C_{10} \cong C_5 \times C_2$ then $KerY$ contains C_2 and it commutes with, C_3, C_5 ,…

This gives that $G \cong C_2 \times N$, where N is a non-abelian group of order 15, which has already been found.

(b) Let $H = D_5$ and D_5 has no quotient group of order 4 , so $Y(D_7)$ has order 1. If $Y(D_5)$ has order 1 then $G \cong D_5 \times C_3$, which has already been found.

10, 3-Sylow subgroups:

The normalized of 3-Sylow subgroup $N(S_3)$ must have subgroup of order 3.

Now the order of $N(S_3) \cong 6$ and so, $N(S_3) \cong 6$ or $C_3 \times C_2$ or D_3 or Δ_3. so, $N(S_3) \cong \Delta_3$, which is a subgroup of order 3.

The possibilities are 1) $N(\Delta_3) = C_6$ 2) $N(\Delta_3) = C_3 \times C_2$

If $N(\Delta_3) = C_6$ then $a^2 = a^3 = e, d^{-1}cd = c^{-1}, d^{-1}ad = a^{-1}, d^2 = a^3, ac = ca$

And $b^3 = e, d^{-1}bd = b^{-1}$ such that $d^2 = b^{15}$ So, $G \cong \Delta_{15}$

5-Sylow Subgroups:

The number x of 7-sylow subgroups of a group G of order 42 is

$x \equiv 1(\bmod ulo 5)$; where $x = 1$.

1, 5-Sylow Subgroups:

Any C_5 is a normal subgroup of G.

The group extension is $\{e\} \to C_3 \to G \to H \to \{e\}$ and H is of order 6. But (5,6)=1, so the extension splits. And so $Y: H \to Aut(C_5) \cong C_6$.

(a) Let $H = C_6 \cong C_3 \times C_2$ then

$KerY$ Contains C_2 and it commutes with, C_5, C_3 ,…. This gives that $G \cong C_2 \times N$, where N is a non-abelian group of Order 15, which has already been found.

(b) Let $H = D_3$ and D_3 has an element of order 5, so Y (D_3) has order 1. If $Y(D_3)$ has order 1 then $G \cong D_3 x C_5$, which is a non-abelian group of order 30

Remarks:

We can list the different groups of order 30 as follows:

Non-abelian groups:

1) $G \cong C_3 \times D_5$ 2) $G \cong D_3 \times C_5$ 3) $G \cong \Delta_{15}$

ALL GROUP OF ORDER 42

(a) Abelian Case

From the experience of the third chapter, we can list the Abelian groups of order 42 as follows:

(1) $G \cong C_{42}$, (2) $G \cong C_2 \times C_3 \times C_7$

(b) Non – Abelian Case

We keep in mind that 42=2.3.7

2- Sylow Subgroups:

The number x of 2-Sylow subgroup of a group G of order 42 is $x \equiv 1(\bmod ulo 2)$, where x= 1, 3,7,21.

1, 2-Sylow subgroup:

It implies that there is a proper normal subgroup in G which may be called N of order 2.

Therefore, $N \cong C_2$.

If $N \cong C_2$, then the sequence of group extension $\{e\} \to N \to G \to C_{21} \to \{e\}$

But, (2, 21) =1, so the extension splits. Now, $Y: C_{21} \to Aut(C_2) \cong Id$ and (21, 2) =1, Where Y is a constant homomorphism and the relation is given by $b^{-1}ab = a^{-1}$ which is a commutative case. So, we exclude this case.

3, 2-Sylow subgroup:

The group G as a permutation on the objects, namely is 3,2-Sylow subgroups. It is a transitive group then the mapping $Y: G \to S_3$ gives that $Y(G) = A_3 or S_3$

i) If $Y(G) = A_3$ then the order of N is 14 and hence $N \cong C_{14}$ or $N \cong C_7 X C_2$ *or* $N \cong D_7$, $N \cong \Delta_7$

Now first two cases are excluded for 2-Sylow subgroups of C_{14} and $C_7 X C_2$ are characteristics.

ii) If $Y(G) = S_3$ then the order of N is 7. $N \cong C_7$.

If $N \cong C_7$ then there is a mapping Z such that $Z: S_3 \to Aut(C_7) = C_6 = C_3 X C_2$.

Hence $Z(S_3) = \{e\} or C_2$.

Now, the group extension is given by $\{e\} \to N = C_3 X C_2 \to G \ {}_1\!\to S_3 \to \{e\}$ where the defining relations of S_3 are $a^3 = b^2 = e$ and $b^{-1}ab = a^{-1}$. If $Z(S_3) = \{e\}$ then $KerZ = S_3$ and so let $N = \{t, u\}$ and $G = \{c, d\}$, where c, d are mapped respectively to a, b. Now $a^3 = b^2 = e, c^3 = e, a = c^{-1}ac, a = b^{-1}ab, b = c^{-1}bc, a = d^{-1}ad, d^{-1}bd = b$ and $d = a^i b, d^{-1}cd = c^{-1}a^i b^k, a^i, b^k \in C_3 x C_2 \in Aut(C_7) = D_7$ Now there exists some S∈G such that $s^2 = b^j$ with j=o, 1 .If $S^2 = b$ then

$a^3 = s^4 = c^3, ac = ca, as = sa, s^{-1}cs = c^{-1}ab, s^2cs^2 = s^{-1}(s^{-1}cs) = ca^{2i}b^{2k} = c$ and $S^2 \in \{a, b\}$ comments with c .This implies that $a^{2i}b^{2k} = e$ with $2i = 0 \pmod 5$ and $2k = 0 \pmod 2$ which implies that i=0, k=0,1 .If k =0 then $s^{-1}cs = c^{-1}$ and if k=1 then $s^{-1}cs = c^{-1}b$ .Note that C_3 generated by a is central because it commutes with every element, put $f = c^2d^2, f^7 = e, d^2 = f^3$ and $d^{-1}cd = c^{-1}$ and so {f, d} generates Δ_7 but C_3 and Δ_7 are normal subgroups and $C_3 \cap \Delta_7 = \{e\}$, hence $G \cong C_3 x \Delta_7$ which is a non –abelian group of 42.If $s^2 = e$ then $G = G_1 x_{s.d} C_2$ where G_1={a ,b} with $a^3 = b^2 = c^3 = e$ and $a = c^{-1}ac, a = b^{-1}ab, c^{-1}bc = b$ and $G \cong C_{21}$.Hence $G \cong C_{21} \underset{s.d}{\times} C_2 = C_7 x (C_3 \underset{s.d}{\times} C_2) \cong C_7 x \Delta_3$ Which is a non-abelian group of order 42.

7, 2- Sylow subgroup:

The normalized of 2-sylow subgroup $N\left(S_2\right)$ must have an invariant subgroup of order 2. Now, the order of $N\left(S_2\right)=6$ and so, $N\left(S_2\right)\cong C_6$ or $C_3\times C_2$ or D_3 but $N(s_2)\neq D_3$ because none of them can have an invariant subgroup of order 2. The possibilities are (1) $N(s_2)=C_6$ (2) $C_7\times\Delta_{21}$ and so by Burnside's Theorem normal 2-complement exists. This will be abelian, so we exclude this case.

21, 2-Sylow Subgroup:

21, 2-Sylow subgroups imply that the order of $N\left(S_2\right)$ is 2.

Now, the normal 2- component N of order 21, given that $N\cong C_{21}$ then group extension is given by $\{e\}\to C_{21}\to G\to H\to\{e\}$ where $H\cong C_2$ .But (2, 21)=1 and so extension splits.

b1. If $H\cong C_{21}$ then $Y:C_2\to Aut(C_{21})\cong C_6XC_2$ and $Y(C_2)=\left\{e\right\}$

b2. If $Y\left(C_2\right)=\left\{e\right\}$ then $G\cong C_{21}XC_2$, which is abelian case and hence we drop it

b3. If $Y\left(C_2\right)=C_2$ then Y(c)=Z

b4. If Y(c)=Z then $c^{-1}ac=a,c^{-1}bc=b,ab=ba$ and this gives that $G\cong G_7xC_3$ which is a non-abelian group of order 42.

3-Sylow Subgroup:

The number x of 3-Sylow subgroups of a group G of order 42 is $X=1(\mathrm{mod}\,ulo3)$; where $x=1,7$

1, 3-Sylow Subgroups:

Any C_3 is a normal subgroup of G.

The group extension is $\left\{e\right\}\to C_3\to G\to H\to\left\{e\right\}$ and H is of order 14. But(3,14)=1, so the extension splits. And so $Y:H\to Aut\left(C_3\right)\cong C_2$.

(a) Let $H=C_{14}\cong C_7\times C_2$ then $KerY$ contains C_2 and it commutes with, $C_3,C_7,\ldots$

This gives that $G\cong C_2\times N$, where N is a non-abelian group of order 21, which has already been found.

(b) Let $H=D_7$ and D_7 has no quotient group of order 6 , so $Y(D_7)$ has order 1.

If $Y(D_7)$ has order 1 then $G \cong D_7 X C_3$, which is a non-abelian group of order 42.

7, 3-Sylow subgroups:

The normalized of 3-Sylow subgroup $N(S_3)$ must have subgroup of order 3. Now the order of $N(S_3)=6$ and so $N(S_3)=C_6 or C_3 X C_2 or D_3 or \Delta_3$ .so, $N(S_3)=\Delta_3$.which is a subgroup of order 3.

The possibilities are 1) $N(\Delta_3)=C_6$ 2) $N(\Delta_3)=C_3 X C_2$

If $N(\Delta_3)=C_6$ then $a^2=a^3=e, d^{-1}cd=c^{-1}, d^{-1}ad=a^{-1}, d^2=a^3, ac=ca$ and $b^3=e, d^{-1}bd=b^{-1}$ such that $d^2=b^{21}$. So, $G \cong \Delta_{21}$

7-Sylow Subgroups:

The number x of 7-sylow subgroups of a group G of order 42 is

$X \equiv 1(\mathrm{mod}\, ulo 7)$; where $X=1$.

1,7-Sylow Subgroups:

Any C_7 is a normal subgroup of G. The group extension is $\{e\} \to C_3 \to G \to H \to \{e\}$ and H is of order 6. But(7,6)=1, so the extension splits. And so $Y: H \to Aut(C_7) \cong C_6$.

(a) Let $H=C_6 \cong C_3 \times C_2$ then

$KerY$ contains C_2 and it commutes with, C_7, C_3 ,…

This gives that $G \cong C_2 \times N$, where N is a non-abelian group of order 21, which has already been found.

(b) Let $H=D_3$ and D_3 has an element of order 7 ,so $Y(D_3)$ has order 1.

If $Y(D_3)$ has order 1 then $G \cong D_3 X C_7$, which is a non-abelian group of order 42.

Remarks:

We can list the different groups of order 42 as follows:

Non-abelian groups:

1) $G \cong G_7 x C_3$	4) $G \cong C_3 x \Delta_7$
2) $G \cong D_7 x C_3$	5) $G \cong C_7 x \Delta_3$
3) $G \cong D_3 x C_7$	6) $G \cong \Delta_{21}$

Conclusion:

We have found all possible subgroups of group of order 30 and 42 by applying P -Sylow's Theorem, which will give us a practical knowledge to see the applications of the Sylow's Theorems.

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